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ON A PARAMETRIZATION OF NON-COMPACT WAVELET MATRICES BY WIENER-HOPF FACTORIZATION

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Abstract. A complete parametrization (one-to-one and onto mapping) of a certain class of non-compact wavelet matrices is introduced in terms of coordinates of infinite-dimensional Euclidian space. The developed method relies on Wiener-Hopf factorization of corresponding unitary matrix functions.

1. INTRODUCTION

Let $l^2(\mathbb{Z})$ be the standard Hilbert space of two-sided sequences of complex numbers. A matrix \mathcal{A} with m rows and infinitely many columns

$$\mathcal{A} = \begin{pmatrix} \cdots & a_{-1}^1 & a_0^1 & a_1^1 & a_2^1 & \cdots \\ \cdots & a_{-1}^2 & a_0^2 & a_1^2 & a_2^2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \\ \cdots & a_{-1}^m & a_0^m & a_1^m & a_2^m & \cdots \end{pmatrix}, \quad a_j^i \in \mathbb{C}, \quad (1)$$

where the rows belong to $l^2(\mathbb{Z})$, is called a wavelet matrix (of rank m) if its rows satisfy the so called *shifted orthogonality condition* [4]:

$$\sum_{k=-\infty}^{\infty} a_{k+mj}^i \overline{a_{k+ms}^r} = \delta_{ir} \delta_{js} \quad \text{for all } 1 \leq i, r \leq m; \quad j, s \in \mathbb{Z} \quad (2)$$

(δ stands for the Kronecker delta). Such matrices are a generalization of ordinary $m \times m$ unitary matrices and they play the crucial role in the theory of wavelets [6] and multirate filter banks [7]. Note that if \mathcal{A} is a wavelet matrix and \mathcal{A}' is obtained by shifting some of its rows by a multiple of m , then \mathcal{A}' is a wavelet matrix as well.

In the *polyphase representation* [8] of matrix \mathcal{A} ,

$$\mathbf{A}(z) = \sum_{k=-\infty}^{\infty} A_k z^k, \quad (3)$$

where $\mathcal{A} = (\dots A_{-1} \ A_0 \ A_1 \ A_2 \ \dots)$ is the partition of \mathcal{A} into $m \times m$ blocks $A_k = (a_{km+j}^i)$, $1 \leq i \leq m$, $0 \leq j \leq m-1$, condition (2) is equivalent to

$$\mathbf{A}(z) \tilde{\mathbf{A}}(z) = I_m, \quad (4)$$

where $\tilde{\mathbf{A}}(z) = \sum_{k=-\infty}^{\infty} A_k^* z^{-k}$ is the *adjoint* of $\mathbf{A}(z)$ ($A^* := \overline{A}^T$ is the Hermitian conjugate, and I_m stands for the $m \times m$ unit matrix). This is easy to see as (2) can be written in the block matrix form

$$\sum_{k=-\infty}^{\infty} A_k A_{l+k}^* = \delta_{l0} I_m.$$

On the other hand, if series (3) is convergent a.e. on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, condition (4) means that \mathbf{A} is a unitary matrix function on the unit circle, i.e.,

$$\mathbf{A}(z) (\mathbf{A}(z))^* = I_m \quad \text{for } z \in \mathbb{T}. \quad (5)$$

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Therefore, wavelet matrices are closely related with unitary matrix functions. There is a natural one-to-one correspondence between them and we will rely on this connection throughout the paper.

Our notion of a wavelet matrix is somewhat different from the standard one. Namely, the *linear condition* $\mathbf{A}(1)\mathbf{e} = \sqrt{m}\mathbf{e}_1$, where $\mathbf{e} = (1, 1, \dots, 1)^T$ and $\mathbf{e}_1 = (1, 0, \dots, 0)^T$, must be satisfied in the usual definition (see [6, Eq. 4.9]) in order the corresponding orthogonal basis of $L^2(\mathbb{R})$ can be constructed by means of \mathbf{A} (see [6, Ch-s 4, 5]). In our consideration, the linear condition is irrelevant. Furthermore, since the structure of coefficients of unitary matrix functions $\mathbf{A}(z)$ and $\mathbf{A}(z) \cdot U$, where U is a constant unitary matrix, are closely related, we introduce the equivalent classes of wavelet matrices as follows:

$$\mathcal{A} \sim \mathcal{A}' \iff A_j = A'_j U \text{ for some constant unitary matrix } U \text{ and every } j \in \mathbb{Z}. \quad (6)$$

We get a unique representative with a corresponding linear condition in each class in this way.

If the number of non-zero columns in (1) is finite, then the wavelet matrix \mathcal{A} is called compact. Otherwise, it is non-compact.

For a compact wavelet matrix

$$\mathbf{A}(z) = \sum_{k=0}^N A_k z^k, \quad (7)$$

in order to avoid a chaotic rearrangement of the rows of \mathcal{A} , we assume that not only $A_0 \neq 0$ and $A_N \neq 0$ (N is called the *order* of (7) in this case) but also

$$\det \mathbf{A}(z) = cz^N. \quad (8)$$

Since it follows from (5) that $\det \mathbf{A}(z)$ is a monomial for compact wavelet matrices, it has necessarily form (8) and the power of z is called the *degree* of (7). It is proved in [1] that the degree of (7) is N if and only if $\text{rank} A_0 = m - 1$ (see Lemma 1 therein). This is the maximal possible value for the rank of A_0 and such situation is naturally called nonsingular.

In [1], a complete parametrization (one-to-one and onto mapping) of compact wavelet matrices of rank m and of order and degree N , with a minor restriction that the last row of A_N is not all zeros (this set is denoted by $\mathcal{CWM}_1[m, N, N]$), is proposed in terms of coordinates in the Euclidian space $\mathbb{C}^{(m-1)N}$. Namely, we have

$$\mathcal{CWM}_1[m, N, N] \longleftrightarrow \underbrace{\mathcal{P}_N^- \times \mathcal{P}_N^- \times \dots \times \mathcal{P}_N^-}_{m-1} \cong \underbrace{\mathbb{C}^N \times \mathbb{C}^N \times \dots \times \mathbb{C}^N}_{m-1} \quad (9)$$

in the following sence: For each $\mathbf{A} \in \mathcal{CWM}_1[m, N, N]$ there exists a unique Laurent matrix polynomial $F(z)$ of the form

$$F(z) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \zeta_1^-(z) & \zeta_2^-(z) & \zeta_3^-(z) & \dots & \zeta_{m-1}^-(z) & 1 \end{pmatrix}, \quad (10)$$

where $\zeta_i^-(z) \in \mathcal{P}_N^-$, $j = 1, 2, \dots, m - 1$, such that

$$F(z)U(z) \in \mathcal{P}_N^+(m \times m),$$

where

$$U(z) = \text{diag}[1, \dots, 1, z^{-N}]\mathbf{A}(z) \quad (11)$$

(the last row of \mathcal{A} is shifted to the left by mN), and

$$\mathcal{P}_N^+ := \left\{ \sum_{k=0}^N c_k z^k : c_k \in \mathbb{C}, k = 0, \dots, N \right\}; \quad \mathcal{P}_N^- := \left\{ \sum_{k=1}^N c_k z^{-k} : c_k \in \mathbb{C}, k = 1, \dots, N \right\}.$$

In other words

$$U(z) = U_-(z)U_+(z),$$

where

$$U_-(z) = F^{-1}(z) \quad \text{and} \quad U_+(z) = F(z)U(z),$$

is the (right) Wiener-Hopf factorization of U . Note that F^{-1} can be obtained from F if we replace each ζ_i^- in (10) by $-\zeta_i^-$.

It readily follows from (11) and properties of \mathbf{A} that the unitary Laurent matrix polynomial U has the following properties:

$$\det U(z) = \text{Const}, \quad \text{and} \quad \sum_{j=1}^m |u_{mj}(0)| > 0.$$

In the present paper, we are going to extend parametrization (9) to a certain class of non-compact wavelet matrices by letting $N \rightarrow \infty$ in the above formulations. To this end, we introduce some additional definitions.

Let $L_p^+ = H_p$, where $0 < p \leq \infty$, be the Hardy space of analytic functions (we usually identify analytic functions in the unit disk and their boundary values on \mathbb{T}) and $L_p^- := \{f : \bar{f} \in L_p^+\}$ be the corresponding set of anti-analytic functions. Denote also

$$L^\pm := \bigcap_{0 < p < \infty} L_p^\pm.$$

Obviously, both of the sets L^+ and L^- are closed under multiplication:

$$f, g \in L^\pm \implies fg \in L^\pm. \quad (12)$$

Let $\mathcal{WM}^\pm[m]$ be the set of equivalent classes (see (6)) of wavelet matrices (1) with $a_j^i = 0$ for $i = 1, 2, \dots, m-1$ and $j < 0$ or $i = m$ and $j \geq m$ (i.e., the entries in the first $m-1$ rows in the polyphase representation (3) are from L_∞^+ and the entries in the last row are from L_∞^-) such that

$$\det \mathbf{A}(z) = \text{Const} \quad \text{for a.a. } z \in \mathbb{T}, \quad (13)$$

and the analytic functions $f_j(z) := \tilde{\mathbf{A}}_{m,j}(z) = \sum_{k=0}^{\infty} \overline{a_{j-1-mk}^m} z^k$, $j = 1, 2, \dots, m$ (the adjoints of the entries in the last row of $\mathbf{A}(z)$) are not simultaneously equal to 0 in the space of maximal ideals of H_∞ , i.e.,

$$\sum_{j=0}^m |f_j(z)| > \delta, \quad |z| < 1, \quad \text{for some } \delta > 0;$$

and let \mathcal{P}_∞^- be the projection of L_∞ on the set of anti-analytic functions vanishing at the infinity, i.e.,

$$\mathcal{P}_\infty^- := \left\{ \sum_{k=-\infty}^{-1} c_k t^k : \text{there exist } f \in L_\infty \text{ such that } \hat{f}(k) = c_k \text{ for } k < 0 \right\} \subset L^-,$$

where $\hat{f}(k)$ stands for the k -th Fourier coefficient of f . Then we have a one-to-one and onto mapping similar to (9):

$$\mathcal{WM}^\pm[m] \longleftrightarrow \underbrace{\mathcal{P}_\infty^- \times \mathcal{P}_\infty^- \times \dots \times \mathcal{P}_\infty^-}_{m-1},$$

which is the claim of the following

Theorem 1. *Let $\mathcal{A} = \mathbf{A}(z) \in \mathcal{WM}^\pm[m]$. Then there exists a unique matrix function $F(z)$ of the form (10), where*

$$\zeta_i^- \in \mathcal{P}_\infty^-, \quad (14)$$

$j = 1, 2, \dots, m-1$, such that

$$F(z)\mathbf{A}(z) \in L^+(m \times m). \quad (15)$$

Conversly, for each matrix function (10), (14) there exists a unique $\mathbf{A}(z) \in \mathcal{WM}^\pm[m]$ such that (15) holds.

The inclusion (15) means again that the representation

$$\mathbf{A}(z) = \mathbf{A}_-(z)\mathbf{A}_+(z),$$

where

$$\mathbf{A}_-(z) = F^{-1}(z) \quad \text{and} \quad \mathbf{A}_+(z) = F(z)\mathbf{A}(z),$$

is the (right) Wiener-Hopf factorization of $\mathbf{A}(z)$.

2. PROOF OF THEOREM 1

Proof of Theorem 1 is based on the technique developed in [2].

Since $\mathbf{A}(z) \in L_\infty(m \times m)$ is a unitary matrix function, we have

$$\mathbf{A}^{-1}(z) = \mathbf{A}^*(z) \quad \text{a.e. on } \mathbb{T}. \quad (16)$$

Because of the Carleson Corona Theorem (see, e.g. [5]) there exist functions g_1, g_2, \dots, g_m from H_∞ such that

$$\sum_{j=1}^m f_j(z)g_j(z) = 1 \quad \text{for } |z| < 1. \quad (17)$$

Let $\mathbf{B} \in L_\infty^+(m \times m)$ be the matrix function \mathbf{A} with its last row replaced by (g_1, g_2, \dots, g_m) . Then, since the last column of \mathbf{A} is $(f_1, f_2, \dots, f_m)^T$ and (16), (17) hold, we have

$$\mathbf{B}\mathbf{A}^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_{m-1} & 1 \end{pmatrix} =: \Phi \in L_\infty(m \times m),$$

where $\zeta_i = \sum_{k=1}^m g_k \tilde{\mathbf{A}}_{ik}$. Thus, it follows from (16) that

$$\Phi\mathbf{A} = \mathbf{B}. \quad (18)$$

Let

$$\zeta_i = \zeta_i^+ + \zeta_i^-, \quad \text{where } \zeta_i^\pm \in \mathcal{P}_\infty^\pm, \quad i = 1, 2, \dots, m-1 \quad (19)$$

(the definition of \mathcal{P}_∞^+ and the inclusion $\mathcal{P}_\infty^+ \subset L^+$ are obvious). Then

$$\Phi = \Phi^+ \Phi^-, \quad (20)$$

where $\Phi^\pm \in \mathcal{P}^\pm$ is the matrix Φ with its last row replaced by $(\zeta_1^\pm, \zeta_2^\pm, \dots, \zeta_{m-1}^\pm, 1)$. The equations (18) and (20) imply that

$$\Phi^- \mathbf{A} = (\Phi^+)^{-1} \mathbf{B} \in L^+(m \times m), \quad (21)$$

which proves (15) if we observe that $F(z) = \Phi^-(z)$ and $(\Phi^+)^{-1}$ is the matrix Φ^+ with its last row replaced by $(-\zeta_1^+, -\zeta_2^+, \dots, -\zeta_{m-1}^+, 1)$.

Let us now prove the uniqueness of F .

Assume

$$F_i(z)\mathbf{A}(z) = \Phi_i^+(z) \in L^+(m \times m), \quad i = 1, 2, \quad (22)$$

are two representations of type (10), (14), where $F_1 = F$ and F_2 is the matrix F with its last row replaced by $(\zeta'_1, \zeta'_2, \dots, \zeta'_{m-1}, 1)$.

Since $\Phi_i^+ \in L^+(m \times m) \implies \det \Phi_i^+ \in L^+$ (see (12)) and $\det \Phi_i^+(z) = C$ a.e. on \mathbb{T} (see (13), (22)), it follows that $\det \Phi_i^+(z) = C$ for $|z| < 1$. Therefore $(\Phi_i^+(z))^{-1} \in L^+(m \times m)$ because of Cramer's formula.

Equations in (22) imply that

$$\mathcal{P}_\infty^-(m \times m) \ni F_2^{-1}(z)F_1(z) = (\Phi_2^+(z))^{-1}\Phi_1^+(z) \in L^+(m \times m).$$

Hence the matrix function $F_2^{-1}F_1$ is constant, while it has form (10) with its last row replaced by $(\zeta_1^- - \zeta'_1, \zeta_2^- - \zeta'_2, \dots, \zeta_{m-1}^- - \zeta'_{m-1}, 1)$. Consequently

$$\zeta_i^- = \zeta'_i \quad \text{for } i = 1, 2, \dots, m-1.$$

Let us now show the converse part of Theorem 1. The essential part of the claim is proved in [3, Lemma 4]: For each matrix of form (10), where $\zeta_i^- \in L_2^-$, $i = 1, 2, \dots, m-1$, there exists a unique (up to a constant right factor) unitary matrix function

$$U(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix}, \quad u_{ij}^+ \in L_\infty^+,$$

with constant determinant

$$\det U(t) = \text{Const} \quad \text{for a.a. } t \in \mathbb{T}, \quad (23)$$

such that

$$F(t)U(t) \in L_2^+(m \times m).$$

It remains to prove that if (14) holds, then

$$\sum_{j=0}^m |u_{mj}^+(z)| > \delta, \quad |z| < 1, \quad \text{for some } \delta > 0, \quad (24)$$

and

$$F(t)U(t) \in L^+(m \times m). \quad (25)$$

We obtain both relations simultaneously.

Since (14) holds, there exist bounded functions $\zeta_i \in L_\infty$ such that (19) holds. Let Φ^\pm be defined as in (20). Then $\Phi^+ F = \Phi^+ \Phi^- = \Phi$ is bounded and therefore

$$\Phi^+ F U =: \Psi^+ \in L_\infty^+(m \times m). \quad (26)$$

Hence

$$F U = (\Phi^+)^{-1} \Psi^+ \in L^+(m \times m)$$

and (25) holds.

To show (24), let us first observe that $\det \Psi^+(z) = \text{Const}$ for $|z| < 1$ since $\det \Psi^+ \in H_\infty$ and it is constant a.e. on the boundary (see (20), (23), and (26)). Therefore

$$\sum_{j=1}^m \Psi_{mi}^+(z) \text{Cof}(\Psi_{mi}^+)(z) = C, \quad (27)$$

where Cof stands for the cofactor. However, the first $m-1$ rows of U and Ψ^+ coincide. So that

$$\text{Cof}(\Psi_{mi}^+) = \text{Cof}(U_{mi}), \quad j = 1, 2, \dots, m. \quad (28)$$

In addition, since U is unitary, i.e., $U^{-1} = U^*$, the formula for the inverse matrix implies that

$$u_{mj}^+ = \frac{1}{C} \text{Cof}(U_{mj}). \quad (29)$$

Therefore, substituting (28) and (29) in (27), we get

$$\sum_{j=1}^m \Psi_{mi}^+(z) u_{mj}^+(z) = 1,$$

and, because of boundedness of the functions Ψ_{mi}^+ (see (26)), relation (24) holds.

3. OPEN PROBLEMS

For compact wavelet matrices, it is proved in [1] that the entries ζ_i^- of the matrix (10) in Theorem 1 can be computed by the formula

$$\zeta_i^-(z) = \mathbb{P}_N^-(\tilde{\mathbf{A}}_{ij}(z)/\mathbf{A}_{mj}(z)), \quad \text{if } \mathbf{A}_{mj}(0) \neq 0, \quad (30)$$

where \mathbb{P}_N^- is the projection of a (formal) Fourier series $\sum_{k=-N}^{\infty} c_k t^k$ on \mathcal{P}_N^- (see [1, Eq. (25)]). To describe the conditions under which we can let $N \rightarrow \infty$ in equation (30) and to determine in which sense the limit exists is an interesting problem. It is related to the computation of partial indices of Wiener-Hopf factorization for a certain class of matrix functions which is the subject of a forthcoming paper.

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